

TOPOLOGICAL ENTROPY AND IE-TUPLES OF INDECOMPOSABLE CONTINUA

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ABSTRACT. In [3], by use of ergodic theory method, Blanchard, Glasner, Kolyada and Maass proved that if a map $f : X \rightarrow X$ of a compact metric space X has positive topological entropy, then there is an uncountable δ -scrambled subset of X for some $\delta > 0$ and hence the dynamics (X, f) is Li-Yorke chaotic. In [18], Kerr and Li developed local entropy theory and gave a new proof of this theorem. Moreover, by developing some deep combinatorial tools, they proved that X contains a Cantor set Z which yields more chaotic behaviors (see [18, Theorem 3.18]). In the paper [6], we proved that if G is any graph and a homeomorphism f on a G -like continuum X has positive topological entropy, then X has an indecomposable subcontinuum. Moreover, if G is a tree, there is a pair of two distinct points x and y of X such that the pair (x, y) is an IE-pair of f and the irreducible continuum between x and y in X is an indecomposable subcontinuum. In this paper, we define a new notion of "freely tracing property by free chains" on G -like continua and we prove that a positive topological entropy homeomorphism on a G -like continuum admits a Cantor set Z such that every tuple of finite points in Z is an IE-tuple of f and Z has the freely tracing property by free chains. Also, by use of this notion, we prove the following theorem: If G is any graph and a homeomorphism f on a G -like continuum X has positive topological entropy, then there is a Cantor set Z which is related to both the chaotic behaviors of Kerr and Li [18] in dynamical systems and composants of indecomposable continua in topology. Our main result is Theorem 3.3 whose proof is also a new proof of [6]. Also, we study dynamical properties of continuum-wise expansive homeomorphisms. In this case, we obtain more precise results concerning continuum-wise stable sets of chaotic continua and IE-tuples.

1. INTRODUCTION

During the last thirty years or so, many interesting connections between dynamical systems and continuum theory have been studied by many authors (see [1,2,6,7,9-15,17,19,22-25,27,28]). We are interested in the following fact that chaotic topological dynamics should imply existence of complicated topological structures of underlying spaces. In many cases, such continua (=compact connected metric spaces) are indecomposable continua which are central subjects of continuum theory in topology. We know that many indecomposable continua often appear as chaotic attractors of dynamical systems. Also, in many cases, the composants of such indecomposable continua are strongly related to stable or unstable (connected) sets of the dynamics. For instance, in the theory of dynamical systems and continuum theory, the Knaster continuum (= Smale's horse shoe), the pseudo-arc, solenoids and Plykin attractors (=Wada's lakes) etc., are well-known as such indecomposable continua.

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In [3], by use of ergodic theory method, Blanchard, Glasner, Kolyada and Maass proved that if a map $f : X \rightarrow X$ of a compact metric space X has positive topological entropy, then there is an uncountable δ -scrambled subset of X for some $\delta > 0$ and hence the dynamics (X, f) is Li-Yorke chaotic. In [18], Kerr and Li developed local entropy theory and gave a new proof of this theorem. Moreover, they proved that X contains a Cantor set Z which yields more chaotic behaviors (see [18, Theorem 3.18]). In [2], Barge and Diamond showed that for piecewise monotone surjections of graphs, the conditions of having positive entropy, containing a horse shoe and the inverse limit space containing an indecomposable subcontinuum are all equivalent. In [24], Mouron proved that if X is an arc-like continuum which admits a homeomorphism f with positive topological entropy, then X contains an indecomposable subcontinuum. In [6], as an extension of the Mouron's theorem, we proved that if G is any graph and a homeomorphism f on a G -like continuum X has positive topological entropy, then X contains an indecomposable subcontinuum. Moreover, if G is a tree, there is a pair of two distinct points x and y of X such that the pair (x, y) is an IE-pair of f and the irreducible continuum between x and y in X is an indecomposable subcontinuum.

In this paper, for any graph G we define a new notion of "freely tracing property by free chains" on G -like continua and by use of this notion, we prove that a positive topological entropy homeomorphism on a G -like continuum admits a Cantor set Z such that every tuple of finite points in Z is an IE-tuple of f and Z has the freely tracing property by free chains. Also, we prove that the Cantor set Z is related to both the chaotic behaviors of Kerr and Li [18] in dynamical systems and composants of indecomposable continua in topology. Our main result is Theorem 3.3 whose proof is also a new proof of [6]. Also, we study dynamical properties of continuum-wise expansive homeomorphisms. In this case, we obtain more precise results concerning continuum-wise stable sets of chaotic continua and IE-tuples.

2. DEFINITIONS AND NOTATIONS

In this paper, we assume that all spaces are separable metric spaces and all maps are continuous. Let \mathbb{N} be the set of natural numbers and \mathbb{Z} the set of integers.

Let X be a compact metric space and \mathcal{U}, \mathcal{V} be two covers of X . Put

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The quantity $N(\mathcal{U})$ denotes minimal cardinality of subcovers of \mathcal{U} . Let $f : X \rightarrow X$ be a map and let \mathcal{U} be an open cover of X . Put

$$h(f, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{\log N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \dots \vee f^{-n+1}(\mathcal{U}))}{n}.$$

The *topological entropy* of f , denoted by $h(f)$, is the supremum of $h(f, \mathcal{U})$ for all open covers \mathcal{U} of X . The reader may refer to [3,4,5,6,8,18,22-25,27,28] for important facts concerning topological entropy. Positive topological entropy of map is one of generally accepted definitions of chaos.

We say that a set $I \subseteq \mathbb{N}$ has *positive density* if

$$\liminf_{n \rightarrow \infty} \frac{|I \cap \{1, 2, \dots, n\}|}{n} > 0.$$

Let X be a compact metric space and $f : X \rightarrow X$ a map. Let \mathcal{A} be a collection of subsets of X . We say that \mathcal{A} has an *independence set with positive density* if there exists a set $I \subset \mathbb{N}$ with positive density such that for all finite sets $J \subseteq I$, and for all $(Y_j) \in \prod_{j \in J} \mathcal{A}$, we have that

$$\bigcap_{j \in J} f^{-j}(Y_j) \neq \emptyset.$$

We observe a simple but important and useful fact that if I is an independence set with positive density for \mathcal{A} then for all $k \in \mathbb{Z}$, $k + I$ is an independence set with positive density for \mathcal{A} . For convenience, we may assume that I satisfies the condition *(kl)*; for all $(Y_j) \in \prod_{j \in J} \mathcal{A}$ and any $Y_0 \in \mathcal{A}$

$$(kl) \ Y_0 \cap \bigcap_{j \in J} f^{-j}(Y_j) \neq \emptyset.$$

We now recall the definition of IE-tuple. Let (x_1, \dots, x_n) be a sequence of points in X . We say that (x_1, \dots, x_n) is an *IE-tuple for f* if whenever A_1, \dots, A_n are open sets containing x_1, \dots, x_n , respectively, we have that the collection $\mathcal{A} = \{A_1, \dots, A_n\}$ has an independence set with positive density. In the case that $n = 2$, we use the term IE-pair. We use IE_k to denote the set of all IE-tuples of length k .

Let $f : X \rightarrow X$ be a map of a compact metric space X with metric d and let $\delta > 0$. A subset S of X is a δ -scrambled set of f if $|S| \geq 2$ and for any $x, y \in S$ with $x \neq y$, then one has

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) \geq \delta.$$

We say that $f : X \rightarrow X$ is *Li-Yorke chaotic* if there is an uncountable subset S of X such that for any $x, y \in S$ with $x \neq y$, then one has

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0.$$

Also, f has *sensitive dependence on initial conditions* if there is a positive number $c > 0$ such that for any $x \in X$ and any neighborhood U of x , one can find $y \in U$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq c$.

Let X_i ($i \in \mathbb{N}$) be a sequence of compact metric spaces and let $f_{i,i+1} : X_{i+1} \rightarrow X_i$ be a map for each $i \in \mathbb{N}$. The *inverse limit* of the inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ is the space

$$\varprojlim \{X_i, f_{i,i+1}\} = \{(x_i)_{i=1}^{\infty} \mid x_i = f_{i,i+1}(x_{i+1}) \text{ for each } i \in \mathbb{N}\} \subset \prod_{i=1}^{\infty} X_i$$

which has the topology inherited as a subspace of the product space $\prod_{i=1}^{\infty} X_i$.

If $f : X \rightarrow X$ is a map, then we use $\varprojlim(X, f)$ to denote the inverse limit of X with f as the bonding maps, i.e.,

$$\varprojlim(X, f) = \left\{ (x_i)_{i=1}^{\infty} \in X^{\mathbb{N}} \mid f(x_{i+1}) = x_i \ (i \in \mathbb{N}) \right\}.$$

Let $\sigma_f : \varprojlim(X, f) \rightarrow \varprojlim(X, f)$ be the *shift homeomorphism* defined by

$$\sigma_f(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

A *continuum* is a compact connected metric space. We say that a continuum is *nondegenerate* if it has more than one point. A continuum is *indecomposable* (see [19,20,23,26])

if it is nondegenerate and it is not the union of two proper subcontinua. For any continuum H , the set $c(p)$ of all points of the continuum H , which can be joined with the point p by a proper subcontinuum of H , is said to be the *composant* of the point $p \in H$ (see [20, p.208]). Note that for an indecomposable continuum H , the following are equivalent;

- (1) the two points p, q belong to same composant of H ;
- (2) $c(p) \cap c(q) \neq \emptyset$;
- (3) $c(p) = c(q)$.

So, we know that if H is an indecomposable continuum, the family

$$\{c(p) \mid p \in H\}$$

of all composants of H is a family of uncountable mutually disjoint sets $c(p)$ which are connected and dense F_σ -sets in H (see [20, p.212, Theorem 6]). Note that a (nondegenerate) continuum X is indecomposable if and only if there are three distinct points of X such that any subcontinuum of X containing any two points of the three points coincides with X , i.e., X is irreducible between any two points of the three points.

Let H be an indecomposable continuum. We say that a subset Z of H is *vertically embedded* to composants of H if no two of points of Z belong to the same composant of H , i.e., if x, y are any distinct points of Z and E is any subcontinuum of H containing x and y , then $E = H$.

A map g from X onto G is an ε -map ($\varepsilon > 0$) if for every $y \in G$, the diameter of $g^{-1}(y)$ is less than ε . A continuum X is *G-like* if for every $\varepsilon > 0$ there is an ε -map from X onto G . For any finite polyhedron G , X is *G-like* if and only if X is homeomorphic to an inverse limit of an inverse sequence of G . Arc-like continua are those which are *G-like* for $G = [0, 1]$. Our focus in this article is on *G-like* continua where G is a graph (= connected 1-dimensional compact polyhedron). A graph G is a *tree* if G contains no simple closed curve. A continuum X is *tree-like* if for any $\varepsilon > 0$ there exist a tree G_ε and an ε -map from X onto G_ε . In this case, G_ε depends on ε . If \mathcal{G} is a collection of subsets of X , then the *nerve* $N(\mathcal{G})$ of \mathcal{G} is the polyhedron whose vertices are elements of \mathcal{G} and there is a simplex $\langle g_1, g_2, \dots, g_k \rangle$ with distinct vertices g_1, g_2, \dots, g_k if

$$\bigcap_i g_i \neq \emptyset.$$

In this paper, we consider the only case that nerves are graphs.

If $\{C_1, \dots, C_n\}$ is a subcollection of \mathcal{G} we call it a *chain* if $C_i \cap C_{i+1} \neq \emptyset$ for $1 \leq i < n$ and $\overline{C_i} \cap \overline{C_j} \neq \emptyset$ implies that $|i - j| \leq 1$. We say that $\{C_1, \dots, C_n\}$ is a *free chain in \mathcal{G}* if it is a chain and, moreover, for all $1 < i < n$ we have that $C \in \mathcal{G}$ with $\overline{C} \cap \overline{C_i} \neq \emptyset$ implies that $C = C_i$, $C = C_{i-1}$ or $C = C_{i+1}$. By the *mesh* of a finite collection \mathcal{G} of sets, we mean the largest of diameters of elements of \mathcal{G} . Note that for a graph G , a continuum X is a *G-like* if and only if for any $\varepsilon > 0$, there is a finite open cover \mathcal{G} of X such that $N(\mathcal{G}) = G$ (which means that $N(\mathcal{G})$ and G are homeomorphic) and the mesh of \mathcal{G} is less than ε . The Knaster continuum (= Smale's horse shoe) and the pseudo-arc are arc-like continua, solenoids are circle-like continua and Plykin attractors are $(S_1 \vee S_2 \vee \dots \vee S_m)$ -like continua, where $S_1 \vee S_2 \vee \dots \vee S_m$ ($m \geq 3$) denotes the one point union of m circles S_i . Such spaces are typical indecomposable continua. The reader may refer to [20] and [26] for standard facts concerning continuum theory.

Let X be a continuum and $m \in \mathbb{N}$. Suppose that A_i ($1 \leq i \leq m$) are m (nonempty) open sets in X and x_i ($1 \leq i \leq m$) are m distinct points of X . We identify the order $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m$ and the converse order $A_m \rightarrow A_{m-1} \rightarrow \cdots \rightarrow A_1$. Then we consider the equivalence class

$$[A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m] = \{A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m; A_m \rightarrow A_{m-1} \rightarrow \cdots \rightarrow A_1\}.$$

Suppose that \mathcal{G} is a finite open cover of X . We say that a chain $\{C_1, \dots, C_n\} \subseteq \mathcal{G}$ follows from the pattern $[A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m]$ if there exist

$$1 \leq k_1 < k_2 < \cdots < k_m \leq n \text{ or } 1 \leq k_m < k_{m-1} < \cdots < k_1 \leq n$$

such that $C_{k_i} \subset A_i$ for each $i = 1, 2, \dots, m$. In this case, more precisely we say that the chain $[C_{k_1} \rightarrow C_{k_2} \rightarrow \cdots \rightarrow C_{k_m}]$ follows from the pattern $[A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m]$. Similarly, we say that a chain $\{C_1, \dots, C_n\} \subseteq \mathcal{G}$ follows from the pattern $[x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m]$ if there exist

$$1 \leq k_1 < k_2 < \cdots < k_m \leq n \text{ or } 1 \leq k_m < k_{m-1} < \cdots < k_1 \leq n$$

such that $x_i \in C_{k_i}$ for each $i = 1, 2, \dots, m$, where

$$[x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m] = \{x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m; x_m \rightarrow x_{m-1} \rightarrow \cdots \rightarrow x_1\}.$$

More precisely, we say that the chain $[C_{k_1} \rightarrow C_{k_2} \rightarrow \cdots \rightarrow C_{k_m}]$ follows from the pattern $[x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m]$.

Let Z be a subset of a G -like continuum X . We say that Z has the *freely tracing property* by (resp. free) chains if for any $\varepsilon > 0$, any $m \in \mathbb{N}$ and any order $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m$ of any m distinct points x_i ($i = 1, 2, \dots, m$) of Z , there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is G and there is a (resp. free) chain in \mathcal{U} which follows from the pattern $[x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m]$.

Example 1. (1) Let $X = [0, 1]$ be the unit interval and D a subset of X . If $|D| \geq 3$, D does not have the freely tracing property by chains.

(2) Let $X = S^1$ be the unit circle and D a subset of X . If $|D| \leq 3$, then D has the freely tracing property by free chains. If $|D| \geq 4$, then D does not have the freely tracing property by chains.

For the case that X is a tree-like, we obtain the following proposition.

Proposition 2.1. *Let X be a tree-like continuum and let D be a subset of X with $|D| \geq 3$. Then the following are equivalent.*

- (1) *For any order $x_1 \rightarrow x_2 \rightarrow x_3$ of three distinct points x_i ($i = 1, 2, 3$) of D and any $\varepsilon > 0$, there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is a tree and there is a chain in \mathcal{U} which follows from the pattern $[x_1 \rightarrow x_2 \rightarrow x_3]$.*
- (2) *D has the freely tracing property by chains; for any $\varepsilon > 0$, any $m \in \mathbb{N}$ and any order $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m$ of any m distinct points x_i ($i = 1, 2, \dots, m$) of D , there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is a tree and there is a chain in \mathcal{U} which follows from the pattern $[x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_m]$.*

- (3) *The minimal continuum H in X containing D is indecomposable and no two of points of D belong to the same composant of H , i.e., D is vertically embedded to composants of H .*

Proof. First, we will show that (1) implies (3). Consider the family \mathcal{K} of all subcontinua of X containing D . Since X is a tree-like continuum, the intersection

$$H = \bigcap \{K \in \mathcal{K}\}$$

is the unique minimal subcontinuum containing D . Suppose, on the contrary, that H is decomposable. Then there are proper subcontinua H_1, H_2 of H with $H = H_1 \cup H_2$. Since H is the minimal continuum containing D , we can choose $x, y \in D$ such that $x \in H_1 - H_2, y \in H_2 - H_1$. Also let z be a point of D with $z \neq x, z \neq y$. We may assume that $z \in H_1$. Choose $\varepsilon > 0$ with $\varepsilon < d(y, H_1)$. By (1), we can choose an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is a tree and there is a chain in \mathcal{U} which follows from the pattern $[x \rightarrow y \rightarrow z]$. Since H_1 is connected, the family

$$\{U \in \mathcal{U} \mid U \cap H_1 \neq \emptyset\}$$

contains a chain from x to z . Then we have a circular chain in \mathcal{U} . Since $N(\mathcal{U})$ is a tree, this is a contradiction. Hence H is indecomposable.

Suppose, on the contrary, that there are two distinct points $x, y \in D$ which are contained in the same composant of H . Choose a proper subcontinuum J of H containing x, y . Let z be any point of D with $z \neq x$ and $z \neq y$. Let $\varepsilon > 0$ be a sufficiently small positive number. By (1), there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is a tree and there is a chain in \mathcal{U} which follows from the pattern $[x \rightarrow z \rightarrow y]$. Since $N(\mathcal{U})$ is a tree and $x, y \in J$, we see that $d(z, J) < \varepsilon$. Since ε is arbitrary small, then we see that $z \in J$. Hence $D \subset J$. Since H is the minimal continuum containing D , this is a contradiction. Consequently, no two of points of D belong to the same composant of H .

Next we will show that (3) implies (1). We assume that the minimal continuum H in X containing D is indecomposable and no two of points of D belong to the same composant of H . Let a, b, c be any three distinct points of D . We consider the order $a \rightarrow b \rightarrow c$. Let $\varepsilon > 0$ be sufficiently small so that $\varepsilon < \min\{d(a, b), d(b, c), d(c, a)\}$. Since X is tree-like, we have a finite open cover \mathcal{V} of X such that the mesh of \mathcal{V} is less than ε , the nerve $N(\mathcal{V})$ of \mathcal{V} is a tree. For any $x \in X$, let V_x be an element of \mathcal{V} containing the point x .

We consider the following cases.

Case(i): V_b separates the vertices V_a from V_c in the nerve $N(\mathcal{V})$.

In this case, we can easily see that there is a chain in \mathcal{V} which follows from the pattern $[a \rightarrow b \rightarrow c]$.

Case(ii): V_b does not separate the vertices V_a from V_c in the nerve $N(\mathcal{V})$.

Since $N(\mathcal{V})$ is a tree, we can choose subfamilies \mathcal{V}' and \mathcal{V}'' of \mathcal{V} such that $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$, $\mathcal{V}' \cap \mathcal{V}'' = \{V_b\}$, the nerves $N(\mathcal{V}')$ and $N(\mathcal{V}'')$ are connected (tree), and $N(\mathcal{V}')$ contains V_a and V_c . Put

$$Y = X - \bigcup \mathcal{V}'',$$

where

$$\bigcup \mathcal{V}'' = \bigcup \{V \mid V \in \mathcal{V}''\}.$$

Consider the component Y_a containing a in Y and the component Y_c containing c in Y . Then we see that $Y_a \cap Y_c = \emptyset$. Suppose, on the contrary, that $Y_a \cap Y_c \neq \emptyset$ and hence $Y_a = Y_c$.

Since X is a tree-like continuum, we see that $H \cap Y_a$ is a proper subcontinuum of H containing a and c . This implies that a and c belong to the same composant of H . This is a contradiction. Hence $Y_a \cap Y_c = \emptyset$. Since Y_a is a component of Y , we can choose a sufficiently small closed and open neighborhood Y_1 of Y_a in Y such that $Y_1 \cap Y_c = \emptyset$. Put $Y_2 = Y - Y_1$. Consider the following families \mathcal{V}_1 and \mathcal{V}_2 of open sets of Y defined by

$$\mathcal{V}_1 = \mathcal{V}'|Y_1, \quad \mathcal{V}_2 = \mathcal{V}'|Y_2,$$

where $\mathcal{V}'|Y_1 = \{V \cap Y_1 \mid V \in \mathcal{V}'\}$. Put

$$\mathcal{U}' = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}''.$$

By use of the cover \mathcal{U}' of X , we can easily construct the desired open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is a tree and there is a chain in \mathcal{U} which follows from the pattern $[a \rightarrow b \rightarrow c]$.

Note that it is trivial that (2) implies (1). Finally, we prove that (1) implies (2). By the induction on m , we prove the implication. In the statement of (2), the case $m = 3$ is the case of (1). We assume that the statement of (2) is true for the case $m \geq 3$. Let $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots \rightarrow x_m \rightarrow x_{m+1}$ be any order of distinct $m + 1$ points x_i ($i = 1, 2, \dots, m + 1$) of D and $\varepsilon > 0$. By induction, there is a finite open cover \mathcal{V} of X such that the mesh of \mathcal{V} is less than ε , the nerve $N(\mathcal{V})$ of \mathcal{V} is a tree and there exists a chain in \mathcal{V} which follows from the pattern $[x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots \rightarrow x_m]$. As above, we consider the following cases. Case(i): V_{x_m} separates the vertices V_{x_i} ($i = 1, 2, \dots, m - 1$) from $V_{x_{m+1}}$ in the nerve $N(\mathcal{V})$. In this case, we can easily see that there is a chain in \mathcal{V} which follows from the pattern $[x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots \rightarrow x_m \rightarrow x_{m+1}]$.

Case(ii): V_{x_m} does not separate the vertices V_{x_i} ($i = 1, 2, \dots, m - 1$) from $V_{x_{m+1}}$ in the nerve $N(\mathcal{V})$.

Since $N(\mathcal{V})$ is a tree, we can choose subfamilies \mathcal{V}' and \mathcal{V}'' of \mathcal{V} such that $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$, $\mathcal{V}' \cap \mathcal{V}'' = \{V_{x_m}\}$, the nerves $N(\mathcal{V}')$ and $N(\mathcal{V}'')$ are connected (tree) and $N(\mathcal{V}')$ contains the vertices V_{x_i} ($i = 1, 2, \dots, m - 1$) and $V_{x_{m+1}}$. As above, we put

$$Y = X - \bigcup \mathcal{V}''.$$

Note that (1) and (3) are equivalent and hence we can use the conditions of (3). By the arguments as above, we can choose a closed and open set Y' of Y containing x_{m+1} such that Y' does not contain any x_i ($i = 1, 2, \dots, m - 1$). Put $Y'' = Y - Y'$. Consider the following families \mathcal{V}_1 and \mathcal{V}_2 of open sets of X defined by

$$\mathcal{V}_1 = \mathcal{V}'|Y', \quad \mathcal{V}_2 = \mathcal{V}'|Y''.$$

Put

$$\mathcal{U}' = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}''.$$

By use of the cover \mathcal{U}' of X , we have the desired open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is a tree and there is a chain in \mathcal{U} which follows from the pattern $[x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots \rightarrow x_m \rightarrow x_{m+1}]$. This completes the proof. \square

3. TOPOLOGICAL ENTROPY ON G -LIKE CONTINUA AND CANTOR SETS WHICH HAVE THE FREELY TRACING PROPERTY BY FREE CHAINS

In [3], by use of ergodic theory method, Blanchard, Glasner, Kolyada and Maass proved that if a map $f : X \rightarrow X$ of a compact metric space X has positive topological entropy, then there is an uncountable δ -scrambled set of f for some $\delta > 0$ and hence the dynamics (X, f) is Li-Yorke chaotic. In [8], Huang and Ye studied local entropy theory and they gave a characterization of positive topological entropy by use of entropy tuples. Moreover, in [18], by use of local entropy theory (IE-tuples), Kerr and Li proved the following more precise theorem.

Theorem 3.1. ([18, Theorem 3.18]) *Suppose that $f : X \rightarrow X$ is a positive topological entropy map on a compact metric space X , and x_1, x_2, \dots, x_m ($m \geq 2$) are finite distinct points of X such that the tuple (x_1, x_2, \dots, x_m) is an IE-tuple of f . If A_i ($i = 1, 2, \dots, m$) is any neighborhood of x_i , then there are Cantor sets $Z_i \subset A_i$ such that the following conditions hold;*

- (1) *every tuple of finite points in the Cantor set $Z = \cup_i Z_i$ is an IE-tuple;*
- (2) *for all $k \in \mathbb{N}$, k distinct points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, one has*

$$\liminf_{n \rightarrow \infty} \max \{d(f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ -scrambled set of f for some $\delta > 0$.

In [6], by use of local entropy theory (IE-tuples), we proved the following theorem.

Theorem 3.2. ([6]) *Suppose that G is any graph and $f : X \rightarrow X$ is a homeomorphism on a G -like continuum X with positive topological entropy. Then X contains an indecomposable subcontinuum. Moreover, if G is a tree, there is a pair of two distinct points x and y of X such that the pair (x, y) is an IE-pair of f and the irreducible continuum between x and y in X is an indecomposable subcontinuum.*

The next theorem is a structure theorem for positive topological entropy homeomorphisms on G -like continua. The result is the main theorem in this paper which implies that for any graph G , a positive topological entropy homeomorphism on a G -like continuum X admits Cantor set Z which yields both some complicated structures in topology and the chaotic behaviors of Kerr and Li [18] in dynamical systems. Especially, the Cantor set Z has the freely tracing property by free chains.

Theorem 3.3. *Let G be any graph, X a G -like continuum and $f : X \rightarrow X$ a homeomorphism on X with positive topological entropy. Suppose that x_1, x_2, \dots, x_m ($m \geq 2$) are finite distinct points of X such that the tuple (x_1, x_2, \dots, x_m) is an IE-tuple of f and A_i ($i = 1, 2, \dots, m$) is any neighborhood of x_i . Then there are Cantor sets $Z_i \subset A_i$ and an indecomposable subcontinuum H of X such that the following conditions hold;*

- (1) *the Cantor set $Z = \cup_{i=1}^m Z_i$ is vertically embedded to composants of H ; i.e., if x, y are distinct points of Z , then the irreducible continuum $Ir(x, y; H)$ between x and y in H is H ,*
- (2) *Z has the freely tracing property by free chains; for any $m \in \mathbb{N}$ and any order $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_m$ of m distinct points x_i ($i = 1, 2, \dots, m$) of Z and any $\varepsilon > 0$, there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is G and there is a free chain in \mathcal{U} which follows from the pattern $[x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_m]$,*
- (3) *every tuple of finite points in the Cantor set Z is an IE-tuple of f , and*
- (4) *for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the*

following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ -scrambled set of f for some $\delta > 0$.

In the statement of Theorem 3.3, we need the condition that X is a G -like continuum for a graph G .

Example 2. Let $g : Z \rightarrow Z$ be a homeomorphism on a Cantor set Z which has positive topological entropy. Let $X = \text{Cone}(Z)$ be the cone of Z and let $f : X \rightarrow X$ be a homeomorphism which is the natural extension of g . Then $h(f) > 0$ and X is tree-like, but X is not G -like for any graph G . Note that X contains no indecomposable subcontinuum. Also, if D is a subset with $|D| \geq 3$, then D does not have the freely tracing property by chains.

We will freely use the following facts from the local entropy theory.

Proposition 3.4. ([18, Propositions, 3.8, 3.9]) *Let X be a compact metric space and let $f : X \rightarrow X$ be a map.*

- (1) *Let (A_1, \dots, A_k) be a tuple of closed subsets of X which has an independent set of positive density. Then, there is an IE-tuple (x_1, \dots, x_k) with $x_i \in A_i$ for $1 \leq i \leq k$.*
- (2) *$h(f) > 0$ if and only if f has an IE-pair (x_1, x_2) with $x_1 \neq x_2$.*
- (3) *IE_k is closed and $f \times \dots \times f$ invariant subset of X^k .*
- (4) *If (A_1, \dots, A_k) has an independence set with positive density and, for $1 \leq i \leq k$, \mathcal{A}_i is a finite collection of sets such that $A_i \subseteq \bigcup \mathcal{A}_i$, then there is $A'_i \in \mathcal{A}_i$ such that (A'_1, \dots, A'_k) has an independence set with positive density.*

To prove Theorem 3.3, we need the following results.

Proposition 3.5. ([6, Proposition 3.1]) *Let $I \subseteq \mathbb{N}$ be a set with positive density and $n \in \mathbb{N}$. Then, there is a finite set $F \subseteq I$ with $|F| = n$ and a positive density set B such that $F + B \subseteq I$.*

Proposition 3.6. ([6, Proposition 3.2]) *Let X be a compact metric space and let $f : X \rightarrow X$ be a map. Let \mathcal{A} be a collection which has an independence set with positive density and $n \in \mathbb{N}$. Then, there is a finite set F with $|F| = n$ such that*

$$\mathcal{A}_F = \left\{ \bigcap_{i \in F} f^{-i}(Y_i) : Y_i \in \mathcal{A} \right\}$$

has an independence set with positive density.

Let $m \geq 2$ and let $\{1, 2, \dots, m\}^n$ be the set of all functions from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, m\}$. For $\sigma \in \{1, 2, \dots, m\}^n$ ($m \geq 2$), we write $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$, where $\sigma(i) \in \{1, 2, \dots, m\}$. Note that $|\{1, 2, \dots, m\}^n| = m^n$.

Proposition 3.7. (cf. [6, Proposition 3.3]) *Let $m, n \in \mathbb{N}$, and $\sigma_1, \dots, \sigma_{[(m-1)n+1][(m-1)^n+1]}$ be any sequence of distinct elements of $\{1, \dots, m\}^n$. Then there are $1 \leq i \leq n$ and*

$$1 \leq k_1 < k_2 < k_3 < \dots < k_m \leq [(m-1)n+1][(m-1)^n+1]$$

such that $\sigma_{k_j}(i) = j$ for $j = 1, \dots, m$.

Proof. First, we prove the following claim (*): If B is a subset of $\{1, 2, \dots, m\}^n$ with $|B| = (m-1)^n + 1$, then there is $1 \leq i \leq n$ such that

$$B(i) = \{1, 2, \dots, m\},$$

where $B(i) = \{\sigma(i) \mid \sigma \in B\}$.

Suppose, on the contrary that for each $1 \leq j \leq n$,

$$|B(j)| \leq m-1,$$

where $B(j) = \{\sigma(j) \mid \sigma \in B\}$. Then we may consider that the set B is a subset of

$$B(1) \times B(2) \times \dots \times B(n)$$

whose cardinality is $\leq (m-1)^n$. This is a contradiction.

Next, we will prove this proposition. We divide the given sequence

$$\sigma_1, \dots, \sigma_{[(m-1)n+1][(m-1)^n+1]}$$

into $(m-1)n+1$ subsequences as follows. Let

$$B_1 = \{\sigma_1, \dots, \sigma_{(m-1)^n+1}\},$$

$$B_2 = \{\sigma_{(m-1)^n+2}, \dots, \sigma_{2[(m-1)^n+1]}\},$$

...

$$B_{(m-1)n+1} = \{\sigma_{[(m-1)n][(m-1)^n+1]+1}, \dots, \sigma_{[(m-1)n+1][(m-1)^n+1]}\}.$$

Since $|B_s| = (m-1)^n + 1$ ($s = 1, 2, \dots, (m-1)n+1$), the above claim (*) implies that for each B_s , there is $1 \leq i_s \leq n$ such that $B_s(i_s) = \{\sigma(i_s) \mid \sigma \in B_s\} = \{1, \dots, m\}$. Define a function $F : \mathcal{B} = \{B_s \mid 1 \leq s \leq (m-1)n+1\} \rightarrow \{1, 2, \dots, n\}$ by $F(B_s) = i_s$. Since $|\mathcal{B}| = (m-1)n+1$, we can find $1 \leq i \leq n$ such that $|F^{-1}(i)| \geq m$. Then we can choose $1 \leq s_1 < s_2 < \dots < s_m \leq (m-1)n+1$ such that $B_{s_j}(i) = \{1, 2, \dots, m\}$ ($j = 1, 2, \dots, m$). By use of this fact, we can choose $\sigma_{k_j} \in B_{s_j}$ such that $\sigma_{k_j}(i) = j$ for $j = 1, \dots, m$. Then

$$1 \leq k_1 < k_2 < \dots < k_m \leq [(m-1)n+1][(m-1)^n+1]$$

and $\sigma_{k_j}(i) = j$ for $j = 1, \dots, m$. □

To check the chaotic behaviors of Kerr and Li ([18, Theorem 3.18]), we need the following lemma.

Lemma 3.8. *Let $f : X \rightarrow X$ be a map of a compact metric space X . Suppose that (A_1, \dots, A_k) is a tuple of closed subsets of X which has an independent set of positive density. Then there is a tuple (A'_1, \dots, A'_k) of closed subsets of X which has an independent set with positive density such that $A'_j \subset A_j$ ($j = 1, 2, \dots, k$), and if $h : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, k\}$ is any function, then there is $n_h \in \mathbb{N}$ such that $f^{n_h}(A'_j) \subset A_{h(j)}$ for each $j = 1, 2, \dots, k$.*

Proof. Suppose that (A_1, \dots, A_k) has an independent set $I \subset \mathbb{N}$ with positive density satisfying the above condition (kl). Let K be the set $\{1, 2, \dots, k\}^k$ of all functions on $\{1, 2, \dots, k\}$. Since $|K| = k^k (= p)$, we can put $K = \{h_1, h_2, \dots, h_p\}$. By Proposition 3.5, there is a finite set $F \subset I$ with $|F| = p$ and a positive density set B such that $F + B \subset I$. Let $F = \{i_1, i_2, \dots, i_p\}$. For each $1 \leq j \leq k$, we put

$$A'_j = A_j \cap \bigcap_{i_s \in F} f^{-i_s}(A_{h_s(j)})$$

Then $\mathcal{A}' = \{A'_j \mid 1 \leq j \leq k\}$ is a desired family. In fact, $f^{i_s}(A'_j) \subset A_{h_s(j)}$ for each $j = 1, 2, \dots, k$. \square

Proposition 3.9. *Let X be a G -like continuum for a graph G and let T be a Cantor set in X . Suppose that T has the freely tracing property by chains. Then any minimal continuum H in X containing T is indecomposable and there is $s \in \mathbb{N}$ such that for any composant c of H , $|c \cap T| \leq s$. In particular, no infinite points of T belong to the same composant of H . Also, there is a subset Z of T such that Z is a Cantor set and Z is vertically embedded to composants of H .*

Proof. For the graph G , we can find sufficiently large $s \in \mathbb{N}$ such that G does not contain s simple closed curves.

Consider the family \mathcal{K} of all subcontinua of X containing T . By the Zorn's lemma, there is a minimal element H of \mathcal{K} . We will show that H is indecomposable. Suppose, on the contrary, that H is decomposable. There are two proper subcontinua A and B of H such that $H = A \cup B$. Since H is a minimal continuum (=irreducible continuum) containing T , there are two points $x, y \in T$ with $x \in A - B, y \in B - A$. Since T is perfect, $T \cap (A - B)$ and $T \cap (B - A)$ are infinite sets. Then there are distinct points $a_i \in T \cap (A - B)$ ($i = 0, 1, 2, \dots, s$), $b_i \in T \cap (B - A)$ ($i = 1, 2, \dots, s$). Let $\varepsilon > 0$ be sufficiently small so that $d(A, \{b_i \mid i = 1, 2, \dots, s\}) > \varepsilon$. Since T has the freely tracing property by chains, there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is G and there is a chain in \mathcal{U} which follows from the pattern

$$[a_0 \rightarrow b_1 \rightarrow a_1 \rightarrow b_1 \rightarrow a_2 \rightarrow b_2 \rightarrow \dots \rightarrow a_{s-1} \rightarrow b_s \rightarrow a_s].$$

Since A is connected, for any two points $z, z' \in A$ there a chain $\{C_1, C_{frm-e}, \dots, C_n\}$ in \mathcal{U} from z to z' such that $C_j \cap A \neq \emptyset$. By use of these facts, we can easily find distinct s simple closed curves in $N(\mathcal{U}) = G$. This is a contradiction. By the similar arguments, we see that for any composant c of H , $|c \cap T| \leq s$. To find a desired Cantor set $Z \subset T$, we consider the following subset of T^2 :

$$R = \{(x, y) \in T^2 \mid \text{there is a proper subcontinuum } F \text{ of } H \text{ with } x, y \in F\}$$

Let $\{U_i \mid i \in \mathbb{N}\}$ be an open base of H and let

$$R_i = \{(x, y) \in T^2 \mid \text{there is a subcontinuum } F \text{ in } H - U_i \text{ with } x, y \in F\}.$$

Note that R_i is a closed set of T^2 and

$$R = \bigcup \{R_i \mid i \in \mathbb{N}\}.$$

Since each composant of H contains no infinite points of T , we see that R is a nowhere dense F_σ -set in T^2 (see [21, p. 71, Application 2]). By [21, p. 70, Corollary 3], there is a Cantor set Z in T such that Z is independent in R , i.e., if $x, y \in Z$ and $x \neq y$, then $(x, y) \notin R$. Then we see that Z is vertically embedded to composants of H . This completes the proof. \square

The following lemma is the key lemma to prove the main theorem.

Lemma 3.10. *Let G be a graph and let f be a homeomorphism on a G -like continuum X with positive topological entropy. Suppose that \mathcal{A} is a finite open collection of X which has an independence set of f with positive density, any distinct elements of \mathcal{A} are disjoint, and $|\mathcal{A}| = m \geq 2$. Then for any $\varepsilon > 0$ and any order $A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_m$ of all elements*

of \mathcal{A} , there exists a finite open cover \mathcal{V} of X satisfying the following conditions;

- (1) the mesh of \mathcal{V} is less than ε ,
- (2) the nerve $N(\mathcal{V})$ of \mathcal{V} is G ,
- (3) for each $A \in \mathcal{A}$ there is a shrink $s(A) \in \mathcal{V}$ with $s(A) \subset A$ such that

$$s(\mathcal{A}) = \{s(A) \mid A \in \mathcal{A}\}$$

has an independence set with positive density, and

- (4) there is a free chain $[s(A_1) \rightarrow s(A_2) \rightarrow \cdots \rightarrow s(A_m)]$ from $s(A_1)$ to $s(A_m)$ in \mathcal{V} which follows from the pattern $[A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m]$.

Proof. We put

$$\mathcal{A} = \{A_1, A_2, \dots, A_m\}.$$

For each $A \in \mathcal{A}$, we can choose an open set $A' \subset \overline{A'} \subset A$ so that

$$\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$$

has an independence set I with positive density (see Proposition 3.4). We choose a sufficiently small positive number $\varepsilon' < \varepsilon$ so that $d(\overline{A'}, X - A) > \varepsilon'$ for each $A \in \mathcal{A}$. Let $E(G)$ be the set of all edges of the graph G and let $|E(G)|$ be the cardinality of the set $E(G)$. We can choose a sufficiently large natural number $n \in \mathbb{N}$ such that

$$m^n > |E(G)| \cdot [(m-1)n+1][(m-1)^n+1].$$

By Proposition 3.5, we have F with $|F| = n$ and B satisfying the condition of Proposition 3.5. Put $F = \{j_1, j_2, \dots, j_n\}$. Recall

$$\mathcal{A}'_F = \left\{ \bigcap_{j \in F} f^{-j}(Y_j) \mid Y_j \in \mathcal{A}' \right\} = \left\{ \bigcap_{i=1}^n f^{-j_i}(A_{\sigma(i)}) \mid \sigma \in \{1, 2, \dots, m\}^n \right\}.$$

Note that $|\mathcal{A}'_F| = m^n$ and \mathcal{A}'_F has the independence set B with positive density. Since F is a finite set, we can choose a sufficiently small positive number $\delta > 0$ such that every pair of distinct elements of \mathcal{A}'_F is at least δ apart and if U is a subset of X whose diameter is less than δ , then the diameter of $f^i(U)$ ($i \in F$) is less than ε' . Since X is G -like, we can choose an open cover \mathcal{U} of X such that $N(\mathcal{U})$ is G and the mesh of \mathcal{U} is less than δ . Since δ is so small, we see that each element of \mathcal{U} intersects at most one element of \mathcal{A}'_F . By Proposition 3.4 (4), we obtain a subcollection \mathcal{U}' of \mathcal{U} such that each element of \mathcal{A}'_F intersects with only one element of \mathcal{U}' , $|\mathcal{U}'| = m^n$ and the family \mathcal{U}' has an independent set of positive density. Then we can choose a free chain \mathcal{C} in \mathcal{U}' such that \mathcal{C} contains at least

$$m^n / |E(G)| \geq [(m-1)n+1][(m-1)^n+1]$$

many elements of \mathcal{U}' . Put $\mathcal{C} = \{C_1, C_2, \dots, C_p\}$. Note that each $U' \in \mathcal{U}'$ determines the element $\sigma \in \{1, 2, \dots, m\}^n$. By Proposition 3.7, we can choose $i \in F$ such that there is a sequence

$$1 \leq k_1 < k_2 < k_3 < \dots < k_m \leq [(m-1)n+1][(m-1)^n+1]$$

such that $C_{k_j} \in \mathcal{U}'$ and

$$f^i(C_{k_j}) \cap A'_j \neq \emptyset$$

for each $j = 1, 2, \dots, m$. By the choice of ε' , $f^i(C_{k_j}) \subset A_j$ for each $j = 1, 2, \dots, m$. Then the free chain

$$[f^i(C_{k_1}) \rightarrow f^i(C_{k_2}) \rightarrow \cdots \rightarrow f^i(C_{k_m})]$$

in $f^i(\mathcal{U})$ follows from the pattern $[A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m]$. Put $s(A_j) = f^i(C_{k_j})$ and $\mathcal{V} = f^i(\mathcal{U})$. Then

$$s(\mathcal{A}) = \{s(A) \mid A \in \mathcal{A}\}$$

is the desired family. This completes the proof. \square

Now, we will prove Theorem 3.3.

Proof. Let $\mathcal{A}_1 = \{A_i \mid i = 1, 2, \dots, m\}$. Then we may assume that \mathcal{A}_1 has an independence set with positive density and the closures of any distinct elements of \mathcal{A}_1 are disjoint. Also, we may assume $|\mathcal{A}_1| = m (= m_1) \geq 3$ and the mesh of \mathcal{A}_1 is less than $\varepsilon_1 \in (0, 1/2)$. By Lemma 3.8, we have a collection

$$\mathcal{A}'_1 = \{A'_1, A'_2, \dots, A'_{m_1}\}$$

of open sets which has an independent set with positive density and satisfies the following condition (KL) for \mathcal{A}'_1 ;

(KL) $A'_i \subset A_i$ ($i = 1, 2, \dots, m_1$), and if $h : \{1, 2, \dots, m_1\} \rightarrow \{1, 2, \dots, m_1\}$ is any function, then there is $n_h \in \mathbb{N}$ such that $f^{n_h}(A'_i) \subset A_{h(i)}$ for each $i = 1, 2, \dots, m_1$.

Consider the set $\mathcal{A}'_1(m_1)$ of orders (=permutations) of all elements of \mathcal{A}'_1 . Note that the cardinality of $\mathcal{A}'_1(m_1)$ is $m_1! = m_1 \cdot (m_1 - 1) \cdots 2 \cdot 1$. We consider the set $\text{Ord } \mathcal{A}'_1(m_1)$ of equivalence classes of elements of $\mathcal{A}'_1(m_1)$, i.e.,

$$\text{Ord } \mathcal{A}'_1(m_1) = \{[A_1^i \rightarrow A_2^i \rightarrow \cdots \rightarrow A_{m_1}^i] \mid i = 1, 2, \dots, q_1\},$$

where $q_1 = m_1!/2$. By Lemma 3.10, there exists a finite open cover \mathcal{U}_1 of X such that the mesh of \mathcal{U}_1 is less than ε_1 and \mathcal{U}_1 satisfies the following conditions;

- (1) the nerve $N(\mathcal{U}_1)$ of \mathcal{U}_1 is homeomorphic to G ,
- (2) for each $A \in \mathcal{A}_1$ there is $s_1(A) \in \mathcal{U}_1$ such that $s_1(A) \subset A' \subset A$, the family

$$\mathcal{A}_1(1) = \{s_1(A) \mid A \in \mathcal{A}_1\}$$

has an independence set with positive density, and we have a free chain

$$[s_1(A_1^1) \rightarrow s_1(A_2^1) \rightarrow \cdots \rightarrow s_1(A_{m_1}^1)]$$

to from $s_1(A_1^1)$ to $s_1(A_{m_1}^1)$ in \mathcal{U}_1 which follows from the pattern

$$[A_1^1 \rightarrow A_2^1 \rightarrow \cdots \rightarrow A_{m_1}^1].$$

This is the case $i = 1$. If we continue this procedure by induction on $i = 1, 2, \dots, q_1$, we obtain a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_{q_1}$ of finite open covers of X and $s_i(A) \in \mathcal{U}_i$ ($A \in \mathcal{A}_1, i = 1, 2, \dots, q_1$) such that the following conditions hold;

- (3) the nerve $N(\mathcal{U}_i)$ of \mathcal{U}_i is homeomorphic to G ,
- (4) \mathcal{U}_{i+1} is a refinement of \mathcal{U}_i ,
- (5) for each $A \in \mathcal{A}_1$, $s_i(A) \in \mathcal{U}_i$ ($i = 1, 2, \dots, q_1$) satisfies that $A \supset A' \supset s_i(A) \supset s_{i+1}(A)$ and the family

$$\mathcal{A}_1(i) = \{s_i(A) \mid A \in \mathcal{A}_1\} \quad (i = 1, 2, \dots, q_1)$$

is an independence set with positive density, and there is a free chain

$$[s_i(A_1^i) \rightarrow s_i(A_2^i) \rightarrow \cdots \rightarrow s_i(A_{m_1}^i)]$$

from $s_i(A_1^i)$ to $s_i(A_{m_1}^i)$ in \mathcal{U}_i follows from the pattern

$$[s_{i-1}(A_1^i) \rightarrow s_{i-1}(A_2^i) \rightarrow \cdots \rightarrow s_{i-1}(A_{m_1}^i)] \quad (i = 1, 2, \dots, q_1),$$

where $s_0(A_j^1) = A_j^1$, etc.

By Proposition 3.6, for each $A \in \mathcal{A}_1(q_1)$, we can choose nonempty open sets $s_{q_1}(A)^+$ and $s_{q_1}(A)^-$ in $s_{q_1}(A)$ such that $\overline{s_{q_1}(A)^+} \cap \overline{s_{q_1}(A)^-} = \emptyset$ and the collection

$$\mathcal{A}_2 = \{s_{q_1}(A)^+, s_{q_1}(A)^- \mid A \in \mathcal{A}_1(q_1)\}$$

has an independence set with positive density.

Let $|\mathcal{A}_2| = m_2 (= 2m_1)$ and $0 < \varepsilon_2 \leq \frac{1}{2} \cdot \varepsilon_1$. By Lemma 3.8, for \mathcal{A}_2 we can choose a collection \mathcal{A}_2' such that the mesh of \mathcal{A}_2' is less than ε_2 and \mathcal{A}_2' satisfies the condition (KL) for \mathcal{A}_2 as above. Also, we consider the set $\mathcal{A}_2'(m_2)$ of permutations of all elements of \mathcal{A}_2' and the set $Ord \mathcal{A}_2'(m_2)$ as above.

By repeated use of Lemma 3.10, we obtain desired families

$$\mathcal{A}_2(i) = \{s_i(A) \mid A \in \mathcal{A}_2\} \quad (i = 1, 2, \dots, q_2)$$

as above, where $q_2 = m_2!/2$. By use of $\mathcal{A}_2(q_2)$, we obtain \mathcal{A}_3 as above. Note that $|\mathcal{A}_3| = m_3 (= 2^2 \cdot m_1)$.

If we continue this procedure, we have a sequence ε_i ($i \in \mathbb{N}$) of positive numbers and sequences of families \mathcal{A}_i and \mathcal{A}_i' of open sets of X satisfying the following conditions;

- (6) $\varepsilon_i > \varepsilon_{i+1}$ ($i \in \mathbb{N}$) and $\lim_{i \rightarrow \infty} \varepsilon_i = 0$,
- (7) the closures of any distinct elements of \mathcal{A}_i are disjoint, each $A \in \mathcal{A}_i$ contains the closures of two elements of \mathcal{A}_{i+1} and the mesh of \mathcal{A}_i is less than ε_i ,
- (8) \mathcal{A}_i and \mathcal{A}_i' have independence sets with positive density,
- (9) \mathcal{A}_i' satisfies the condition (KL) for \mathcal{A}_i , and
- (10) for any order $E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{m_i}$ of all elements of \mathcal{A}_i , there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε_i , $N(\mathcal{U})$ is G and there is a free chain in \mathcal{U} which follows from

$$[E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{m_i}].$$

For each $i \in \mathbb{N}$, we put

$$T_i = \bigcup \mathcal{A}_i \quad (i \in \mathbb{N}) \quad \text{and}$$

$$T = \bigcap_{i \in \mathbb{N}} T_i.$$

Then T is a Cantor set. By the above constructions, we see that for any $k \in \mathbb{N}$ and any order

$$x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k$$

of k distinct points x_i ($i = 1, 2, \dots, k$) of T and any $\varepsilon > 0$, there is an open cover \mathcal{U} of X such that the mesh of \mathcal{U} is less than ε , the nerve $N(\mathcal{U})$ of \mathcal{U} is G and there is a free chain in \mathcal{U} which follows from the pattern

$$[x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k].$$

By Proposition 3.9, any minimal continuum H in X containing T is indecomposable and no infinite points of T belong to the same composant of H . Also, by Proposition 3.9, we can choose a subset Z of T such that Z is a Cantor set and Z is vertically embedded to composants of H . Also, by the constructions, we see that T satisfies the conditions (2),

(3) and (4) of Theorem 3.3. Note that any subset of T satisfies the conditions. Hence the Cantor set Z satisfies the conditions of Theorem 3.3. This completes the proof. \square

Corollary 3.11. *Let G be any graph. If $f : G \rightarrow G$ is a positive entropy map on G , then there exist an indecomposable subcontinuum H of $X = \varprojlim(G, f)$ and a Cantor set Z in H satisfies the following conditions;*

- (1) Z is vertically embedded to composants of H ,
- (2) Z has the freely tracing property by free chains,
- (3) every tuple of finite points in the Cantor set Z is an IE-tuple of the shift map σ_f and
- (4) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(\sigma_f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ -scrambled set of σ_f for some $\delta > 0$.

Proof. Note that $h(f) = h(\sigma_f) > 0$ and σ_f is a homeomorphism on the G -like continuum $X = \varprojlim(G, f)$. This result follows from Theorem 3.3. \square

For a special case, we have the following.

Corollary 3.12. *Let X be one of the Knaster continuum, solenoids or Plykin attractors. If f is any positive topological entropy homeomorphism on X , then there is a Cantor set Z in X such that the Cantor set Z satisfies the following conditions;*

- (1) Z is vertically embedded to composants of X ,
- (2) Z has the freely tracing property by free chains,
- (3) every tuple of finite points in the Cantor set Z is an IE-tuple of f , and
- (4) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ -scrambled set of f for some $\delta > 0$.

Proof. By Theorem 3.3, there is an indecomposable subcontinuum H in X . Note that any proper subcontinuum of X is not indecomposable and hence $H = X$. This corollary follows from Theorem 3.3. \square

An onto map $f : X \rightarrow Y$ of continua is *monotone* if for any $y \in Y$, $f^{-1}(y)$ is connected. In [16], we proved that if G is a graph and $f : X \rightarrow X$ is a monotone map on a G -like continuum X which has positive topological entropy, then X contains an indecomposable subcontinuum. Here we give the following more precise result.

Theorem 3.13. *Suppose that G is a graph and X is a G -like continuum. If $f : X \rightarrow X$ is a monotone map on X with positive topological entropy, then there exist an indecomposable subcontinuum H of X and a Cantor set Z in H such that the Cantor set Z satisfies the following conditions;*

- (1) Z is vertically embedded to composants of H ,
- (2) every tuple of finite points in the Cantor set Z is an IE-tuple of f ,

(3) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ -scrambled set of f for some $\delta > 0$.

A continuum E is an n -od ($2 \leq n < \infty$) if E contains a subcontinuum A such that the complement of A in E is the union n nonempty mutually separated sets, i.e.,

$$E - A = \bigcup \{E_i \mid i = 1, 2, \dots, n\}$$

for some subsets E_i satisfying the condition:

$$\overline{E_i} \cap E_j = \emptyset \quad (i \neq j).$$

For any continuum X , let

$$T(X) = \sup\{n \mid \text{there is an } n\text{-od in } X\}.$$

Note that if X is a G -like continuum for a graph G , then $T(X) < \infty$.

To prove Theorem 3.13, we need the following lemma.

Lemma 3.14. (cf. [16, Lemma 2.3]) *Let X and Y be continua with $T(X) < \infty$. Suppose that $f : X \rightarrow Y$ is an (onto) monotone map, H' is an indecomposable subcontinuum of X and Z' is a Cantor set which is vertically embedded to composants of H' . If $H = f(H')$ is nondegenerate, then H is an indecomposable subcontinuum of Y and there is a subset Z of $f(Z')$ such that Z is a Cantor set and Z is vertically embedded to composants of H .*

Proof. Note that if $f : X \rightarrow Y$ is monotone, then $f^{-1}(C)$ is connected for any subcontinuum C in Y . By use of this fact, we can see that $T(Y) \leq T(X) < \infty$. For each $x \in Z'$, let $c(x)$ denote the composant of H' containing $x \in Z'$. Let

$$\text{Comp}(Z'; H') = \{c(x) \mid x \in Z'\}.$$

Since Z' is vertically embedded to composants of H' , $\text{Comp}(Z'; H')$ is a family of mutually disjoint dense connected subsets $c(x)$ of H' . For each $x, y \in Z'$, we define $x \sim_f y$ provided that $f(c(x)) \cap f(c(y)) \neq \emptyset$. Also, we define $x \sim y$ provided that there is a finite sequence $x = x_1, x_2, \dots, x_s = y$ of $x_i \in Z'$ such that $x_i \sim_f x_{i+1}$ for each $i = 1, 2, \dots, s-1$. Then the relation \sim is an equivalence relation on Z' . Note that $f(c(x)) \cap f(c(y)) \neq \emptyset$ if and only if there is a point $z \in Y$ with

$$f^{-1}(z) \cap c(x) \neq \emptyset \neq f^{-1}(z) \cap c(y).$$

Let $[x]$ denote the equivalence class containing $x \in Z'$, i.e.,

$$[x] = \{y \in Z' \mid x \sim y\}.$$

Since $f^{-1}(z)$ is a subcontinuum of X for each $z \in Y$, we can conclude that $|[x]| \leq T(X)$. In particular, $f|_{Z'} : Z' \rightarrow f(Z')$ is a finite-one map and hence $f(Z')$ is a perfect set, i.e., $f(Z')$ has no isolated point.

Since Z' is an uncountable set, we can choose an uncountable subset Z'' of Z' such that the family

$$\{f(c(x)) \mid x \in Z''\}$$

is a family of mutually disjoint subsets of $H = f(H')$.

We will prove that $H = f(H')$ is indecomposable. Suppose, on the contrary, that H is decomposable. There is a proper subcontinuum A of $H = f(H')$ with

$$\text{Int}_H(A) \neq \emptyset.$$

Since each composant of H' is dense in H' and hence $f(c(x))$ is dense in H for any $x \in Z'$, $f(c(x)) \cap A \neq \emptyset$. This implies that $|T(Y)| = \infty$. This is a contradiction. Hence $H = f(H')$ is indecomposable.

We show that for each composant c of H , $|c \cap f(Z')| \leq T(X)$. Suppose, on the contrary, that there is a proper subcontinuum C of H such that $|C \cap f(Z')| \geq T(X) + 1$. Then $f^{-1}(C)$ is a continuum which intersects $T(X) + 1$ composants of H' . This is a contradiction.

Since $f(Z')$ is perfect, by the proof of Proposition 3.9, we can find a Cantor set Z in $f(Z')$ such that Z is vertically embedded to composants of H . \square

We will give the proof of Theorem 3.13.

Proof. We consider the inverse $\tilde{f} : \varprojlim(X, f) \rightarrow \varprojlim(X, f)$ of the shift map σ_f , i.e.,

$$\tilde{f}(x_1, x_2, x_3, \dots) = (f(x_1), x_1, x_2, \dots).$$

Note that $h(f) = h(\tilde{f}) > 0$. By Theorem 3.3, we can find an indecomposable subcontinuum H' and a Cantor set Z' in $\varprojlim(X, f)$ as in Theorem 3.3. Since f is a monotone map, we see that the projection $p_n : \varprojlim(X, f) \rightarrow X_n = X$ to the n -th coordinate X_n is also monotone. If we choose sufficiently large natural number n , then $H = p_n(H')$ is nondegenerate. By the above lemma, H is indecomposable and there is a Cantor set $Z \subset p_n(Z')$ such that Z is vertically embedded to composants of H . Note that the projection p_n preserves the properties of IE-tuples and (4) of Theorem 3.3. Then we see that H and Z are desired spaces. \square

4. CHAOTIC CONTINUA OF CONTINUUM-WISE EXPANSIVE HOMEOMORPHISMS AND IE-TUPLES

In this section, we study dynamical behaviors of continuum-wise expansive homeomorphisms related to IE-tuples and chaotic continua in topology. Any continuum-wise expansive homeomorphism f on a continuum X has positive topological entropy and hence f has IE-tuples (see Theorem 4.1 below). Also, X contains a chaotic continuum and chaotic continuum has uncountable mutually disjoint (unstable or) stable dense connected F_σ -sets (see Theorem 4.1). In this section, we study some precise results of IE-tuples related to (unstable) stable connected sets of chaotic continua and composants of indecomposable continua.

A homeomorphism $f : X \rightarrow X$ of a compact metric space X with metric d is called *expansive* ([5, 13]) if there is $c > 0$ such that for any $x, y \in X$ and $x \neq y$, then there is an integer $n \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism $f : X \rightarrow X$ of a compact metric space X is *continuum-wise expansive* (resp. *positively continuum-wise expansive*) [15] if there is $c > 0$ such that if A is a nondegenerate subcontinuum of X , then there is an integer $n \in \mathbb{Z}$ (resp. a positive integer $n \in \mathbb{N}$) such that

$$\text{diam } f^n(A) > c,$$

where $\text{diam} B = \sup\{d(x, y) \mid x, y \in B\}$ for a set B . Such a positive number c is called an *expansive constant* for f . Note that each expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (see [15]). These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (see [5, 10-15, 27]).

The hyperspace 2^X of X is the set of all nonempty closed subsets of X with the Hausdorff metric d_H . Let

$$C(X) = \{A \in 2^X \mid A \text{ is connected}\}.$$

Note that 2^X and $C(X)$ are compact metric spaces (e.g., see [20] and [26]). For a homeomorphism $f : X \rightarrow X$ and for each closed subset H of X and $x \in H$, the *continuum-wise σ -stable sets* $V^\sigma(x; H)$ ($\sigma = s, u$) of f are defined as follows:

$$V^s(x; H) = \{y \in H \mid \text{there is } A \in C(H) \text{ such that } x, y \in A \text{ and } \lim_{n \rightarrow \infty} \text{diam } f^n(A) = 0\},$$

$$V^u(x; H) = \{y \in H \mid \text{there is } A \in C(H) \text{ such that } x, y \in A \text{ and } \lim_{n \rightarrow \infty} \text{diam } f^{-n}(A) = 0\}.$$

Note that

$$V^s(x; H) \subset W^s(x) = \{y \in X \mid \lim_{n \rightarrow \infty} d(f^n(y), f^n(x)) = 0\},$$

$$V^u(x; H) \subset W^u(x) = \{y \in X \mid \lim_{n \rightarrow \infty} d(f^{-n}(y), f^{-n}(x)) = 0\}.$$

A subcontinuum H of X is called a *σ -chaotic continuum* (see [13]) of f (where $\sigma = s, u$) if

- (1) for each $x \in H$, $V^\sigma(x; H)$ is dense in H , and
- (2) there is $\tau > 0$ such that for each $x \in H$ and each neighborhood U of x in X , there is $y \in U \cap H$ such that

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) \geq \tau \text{ in case } \sigma = s, \text{ or}$$

$$\liminf_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) \geq \tau \text{ in case } \sigma = u.$$

We know that if $f : X \rightarrow X$ is a continuum-wise expansive homeomorphism, then $V^\sigma(z; H)$ is a connected F_σ -set containing z . If H is a σ -chaotic continuum of f , then the decomposition $\{V^\sigma(z; H) \mid z \in H\}$ of H is an uncountable family of mutually disjoint, dense connected F_σ -sets in H . Note that σ -chaotic continua of f have very similar structures of composants of indecomposable continua. In fact, for the case of 1-dimensional continua, σ -chaotic continua may be indecomposable (see [10]).

Example 3. Let $f : T^2 \rightarrow T^2$ be an Anosov diffeomorphism on the 2-dimensional torus T^2 , say

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Then f is expansive and T^2 itself is a σ -chaotic continuum of f for $\sigma = u, s$. Note that T^2 contains no indecomposable σ -chaotic subcontinuum.

For continuum-wise expansive homeomorphisms, we have obtained the following results (see [11, 13, 15]).

Theorem 4.1. *Let $f : X \rightarrow X$ be a continuum-wise expansive homeomorphism on a continuum X . Then the followings hold.*

- (1) ([15, Theorem 4.1]) f has positive topological entropy and hence there are IE-tuples.
- (2) ([13, Theorem 3.6 and Theorem 4.1]) There is a σ -chaotic continuum H of f . Moreover, if H is a u -chaotic continuum (resp. s -chaotic continuum), then there exists a Cantor set Z in H satisfying the conditions;
 - (i) no two of points of Z belong to the same $V^u(x; H)$ ($x \in X$) (resp. $V^s(x; H)$ ($x \in X$)), i.e., Z is vertically embedded to $V^\sigma(x, H)$ ($x \in H$),
 - (ii) Z is a δ -scrambled set of f^{-1} for some $\delta > 0$ (resp. f).
- (3) ([11, Theorem 2.4]) Moreover, if $f : X \rightarrow X$ is a positively continuum-wise expansive homeomorphism, then X contains a u -chaotic continuum H such that H is indecomposable and the set of composants of H coincides to $\{V^u(x; H) \mid x \in H\}$. Also, there exists a Cantor set Z in H satisfying the conditions;
 - (i) Z is vertically embedded to composants $V^u(x, H)$ ($x \in H$),
 - (ii) Z is a δ -scrambled set of f^{-1} for some $\delta > 0$.
- (4) (d) ([11, Corollary 2.7]) Moreover, if G is any graph and X is a G -like continuum, then X contains a σ -chaotic continuum H such that H is indecomposable and the set of composants of H coincides to $\{V^\sigma(x; H) \mid x \in H\}$. Moreover if $\sigma = u$ (resp. s), then there exists a Cantor set Z in H satisfying the conditions;
 - (i) Z is vertically embedded to $V^\sigma(x, H)$ ($x \in H$),
 - (ii) Z is a δ -scrambled set of f^{-1} for some $\delta > 0$ (resp. f).
- (5) ([11, Theorem 2.6]) Moreover, if X is a continuum in the plane \mathbb{R}^2 , then X contains a σ -chaotic continuum H of f such that H is indecomposable and the set of composants of H coincides to $\{V^\sigma(x, H) \mid x \in H\}$. Moreover if $\sigma = u$ (resp. s), then there exists a Cantor set Z in H satisfying the conditions;
 - (i) Z is vertically embedded to $V^\sigma(x, H)$ ($x \in H$),
 - (ii) Z is a δ -scrambled set of f^{-1} for some $\delta > 0$ (resp. f).

We consider the case that σ -chaotic continua are periodic. By combining Theorem 3.1 and Theorem 4.1, we have the following results.

Corollary 4.2. *Let $f : X \rightarrow X$ be a continuum-wise expansive homeomorphism on a continuum X . Suppose that X contains a periodic σ -chaotic continuum H of f . Then there exists a Cantor set Z in H such that if $\sigma = u$ (resp. $\sigma = s$), then the following conditions hold;*

- (1) Z is vertically embedded to $V^\sigma(x, H)$ ($x \in H$),
- (2) every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1} (resp. f), and
- (3) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(f^{-n}(y_i), z_i) \mid 1 \leq i \leq k\} = 0$$

$$(\text{resp. } \liminf_{n \rightarrow \infty} \max\{d(f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0).$$

Proof. We may assume $\sigma = s$. Since the chaotic continuum H is periodic, there is $i \in \mathbb{N}$ such that $f^i(H) = H$. Then $f^i|_H : H \rightarrow H$ is continuum-wise expansive and hence its topological entropy is positive. By Theorem 3.1, there is a Cantor set Z in H as in Theorem

3.1. Since Z is a δ -scrambled set of f for some $\delta > 0$, Z is vertically embedded to $V^s(x, H)$ ($x \in H$). \square

Similarly, we have the following result.

Corollary 4.3. *Suppose that $f : X \rightarrow X$ is a positively continuum-wise expansive homeomorphism on a continuum X such that X has a periodic u -chaotic continuum H which is indecomposable and the set of composants of H coincides to $\{V^u(x; H) \mid x \in H\}$. Then there exists a Cantor set Z in H which is vertically embedded to composants of H and satisfies the conditions;*

- (1) *if x, y belong to the same composant of H , then $\lim_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) = 0$,*
- (2) *every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1} , and*
- (3) *for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds*

$$\liminf_{n \rightarrow \infty} \max\{d(f^{-n}(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

For special cases, we have the following.

Corollary 4.4. *Suppose that X is one of the Knaster continuum, Plykin attractors or solenoids. If $f : X \rightarrow X$ is a continuum-wise expansive homeomorphism on X , then f or f^{-1} is positively continuum-wise expansive. In particular, if f is positively continuum-wise expansive, then there exists a Cantor set Z in X such that the Cantor set Z is vertically embedded to composants of X and satisfies the conditions;*

- (1) *if x, y belong to the same composant of X , then $\lim_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) = 0$,*
- (2) *every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1} ,*
- (3) *Z has the freely tracing property by free chains, and*
- (4) *for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds*

$$\liminf_{n \rightarrow \infty} \max\{d(f^{-n}(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

Proof. Note that X is a G -like continuum for some graph G . Recall that X is indecomposable and each proper subcontinuum of X is not indecomposable. Hence a σ -chaotic continuum of f coincides with X . Then we can easily see that f or f^{-1} is positively continuum-wise expansive. The corollary follows from Theorems 3.3 and 4.1. \square

For the case of the shift map $\sigma_f : \varprojlim(G, f) \rightarrow \varprojlim(G, f)$ of a map $f : G \rightarrow G$ on a graph G which has sensitive dependence on initial conditions, we can find a periodic indecomposable s -chaotic continuum in $\varprojlim(G, f)$. Hence we have the following corollary.

Corollary 4.5. *Suppose that $f : G \rightarrow G$ is a map on a graph G which has sensitive dependence on initial conditions and $\sigma_f : X = \varprojlim(G, f) \rightarrow X$ is the shift map of f . Then there exists an indecomposable s -chaotic continuum H in X such that $\sigma_f^n(H) = H$ for some $n \in \mathbb{N}$ and the set of composants of H coincide to $\{V^s(x; H) \mid x \in H\}$. Hence there is a Cantor set Z in H such that Z is vertically embedded to composants of H and satisfies the conditions;*

- (1) *if x, y belong to the same composant of H , then $\lim_{n \rightarrow \infty} d(\sigma_f^n(x), (\sigma_f^n(y))) = 0$,*
- (2) *every tuple of finite points in the Cantor set Z is an IE-tuple of σ_f ,*

- (3) Z has the freely tracing property by free chains, and
 (4) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(\sigma_f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

Proof. Note that $(\sigma_f)^{-1} = \tilde{f}$. By [11, Corollary 2.8], there is a s -chaotic indecomposable continuum H of σ_f and a natural number n such that $\sigma_f^n(H) = H$ and the set of composants of H coincides to $\{V^s(x, H) \mid x \in H\}$. By use of Theorem 3.3, we can find a desired Cantor set Z in H . \square

Example 4. Let $f : I = [0, 1] \rightarrow I$ be the map defined by $f(t) = 4t(1-t)$ ($t \in I$). Note that f has sensitive dependence on initial conditions and $\varprojlim(X, f)$ is the Knaster continuum. Then $\varprojlim(X, f)$ is the s -chaotic continuum of the shift homeomorphism $\sigma_f : \varprojlim(X, f) \rightarrow \varprojlim(X, f)$ satisfying the conditions of Corollary 4.5, where $H = \varprojlim(X, f)$.

For the general case that any σ -chaotic continua of a continuum-wise expansive homeomorphism are not periodic, we do not know if the statements of Corollaries 4.2 and 4.3 hold. In fact, the following problems remain open.

Question 4.6. Let $f : X \rightarrow X$ be a continuum-wise expansive homeomorphism on a continuum X . Is it true that there exist a σ -chaotic continuum H of f and a Cantor set Z in H such that Z is vertically embedded to $V^\sigma(x, H)$ ($x \in H$) and satisfies the following conditions? :

If $\sigma = u$ (resp. $\sigma = s$), then

- (1) every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1} (resp. f), and
 (2) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(f^{-n}(y_i), z_i) \mid 1 \leq i \leq k\} = 0$$

(resp. $\liminf_{n \rightarrow \infty} \max\{d(f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0$).

Question 4.7. Let $f : X \rightarrow X$ be a positively continuum-wise expansive homeomorphism on a continuum X . Is it true that there exist an indecomposable continuum H and a Cantor set Z in H satisfying the following conditions? :

- (1) Z is vertically embedded to composants of H .
 (2) If x, y belong to the same composant of H , then $\lim_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) = 0$.
 (3) Every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1} .
 (4) For all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, \dots, y_k \in Z$ and any points $z_1, z_2, \dots, z_k \in Z$, the following condition holds

$$\liminf_{n \rightarrow \infty} \max\{d(f^{-n}(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

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